Exercises on the Polynomial Hierarchy PH
CSCI 6114 Fall 2023

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September 12, 2023

Definition 1. The *complement* of a language $L \subseteq \Sigma^*$ is $\overline{L} := \{x \in \Sigma^* : x \notin L\}$. If $\mathcal{C}$ is a collection of languages, then $\mathsf{co}\mathcal{C} = \{\overline{L} : L \in \mathcal{C}\}$.

1. (a) Show that for any complexity class $\mathcal{C}$, $\mathcal{C} \subseteq \mathsf{co}\mathcal{C}$ iff $\mathcal{C} = \mathsf{co}\mathcal{C}$.
   (b) Show that $\mathsf{P} = \mathsf{coP}$. (We say that P "is closed under complement.")
   (c) Show that for any $\mathcal{C}$, $\mathsf{co}\mathcal{C} \cup \mathcal{C}$ and $\mathsf{co}\mathcal{C} \cap \mathcal{C}$ are closed under complement.

2. Is $\mathsf{NP} = \mathsf{coNP}$? This is a hard problem. Try to convince each other one way or the other.

3. (a) Show that, for any oracle X, P^X is closed under complement, that is, $\mathsf{P}^X = \mathsf{coP}^X$. In particular, $\mathsf{P}^{\mathsf{NP}} = \mathsf{coP}^{\mathsf{NP}}$.
   (b) Show that $\mathsf{P}^{\mathsf{NP}} = \mathsf{P}^{\mathsf{coNP}}$.
   (c) Show that if $\mathsf{NP} = \mathsf{P}^{\mathsf{NP}}$, then $\mathsf{NP} = \mathsf{coNP}$.

Definition 2. We use $\exists^p, \forall^p$ to denote the polynomially-bounded version of these quantifiers.

For example, we can (re)define NP as the class of languages L such that there is a polynomial-time verifier V, and for all x,

$$\begin{aligned} x \in L &\iff (\exists^p y)[V(x,y) = 1] \\ &\iff (\exists y)[|y| \leq \text{poly}(|x|) \text{ and } V(x,y) = 1] \end{aligned}$$

Definition 3. 1. A language L is in $\mathsf{\Sigma_k P}$ ($k \geq 0$) if there is a polynomial-time verifier V such that, for all x,

$$x \in L \iff (\exists^p y_1)(\forall^p y_2) \cdots (\exists^p / \forall^p y_k) V(x, y_1, y_2, \ldots, y_k) = 1.$$

where the final quantifier is $\exists^p$ if k is odd and $\forall^p$ if k is even.

2. We similarly define $\mathsf{\Pi_k P}$ except where the right-hand side starts with $\forall^p y_1$ (and then alternate).

3. Finally, we define $\mathsf{PH} = \bigcup_{k \geq 0} \mathsf{\Sigma_k P}$.

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4. Show that $\mathsf{P} = \mathsf{\Sigma_0 P} = \mathsf{\Pi_0 P}$, $\mathsf{NP} = \mathsf{\Sigma_1 P}$, and $\mathsf{coNP} = \mathsf{\Pi_1 P}$.

5. (a) Show that $\mathsf{PH} \subseteq \mathsf{EXP}$, where EXP is the class of decision problems that can be decided by a Turing machine that runs in time $2^{\text{poly}(n)}$.

   (b) Show that $\mathsf{PH} \subseteq \mathsf{PSPACE}$, where PSPACE is the class of decision problems that can be decided by a Turing machine that uses an amount of *space* that is poly(n) (with no *a priori* upper bound on its runtime).

6. Show that $\mathsf{\Sigma_k P} = \mathsf{co\Pi_k P}$. That is, $L \in \mathsf{\Sigma_k P}$ iff $\overline{L} \in \mathsf{\Pi_k P}$ ($\overline{L}$ is our notation for the complement language, $\overline{L} := \Sigma^* \backslash L = \{x \in \Sigma^* | x \notin L\}$). If this feels too abstract, start with $k = 1$.

7. Show that $\mathsf{\Sigma_k P} \subseteq \mathsf{\Sigma_{k+1} P} \cap \mathsf{\Pi_{k+1} P}$. Conclude that (a) $\mathsf{\Sigma_k P} \cup \mathsf{\Pi_k P} \subseteq \mathsf{\Sigma_{k+1} P} \cap \mathsf{\Pi_{k+1} P}$, (b) $\mathsf{PH} = \bigcup_{k \geq 0} \mathsf{\Pi_k P}$.

Resources

- Defined in Stockmeyer, *Theoret. Comp. Sci.*, 1976
- Arora & Barak Ch. 5
- Du & Ko Ch. 3
- Schöning & Pruim, *Gems of TCS*, Ch. 16
- Hemaspaandra & Ogihara, *Complexity Theory Companion*, Appendix A.4.1
- Homer & Selman §7.4 do PH in terms of oracles; we'll see that characterization later, so I'm including it here for future reference, but we haven't gotten to it yet.